RESEARCHER POSITIONING IN THE DIDACTIC TRANSPOSITION PROCESS: EPISTEMOLOGICAL PERSPECTIVES ON MATHEMATICAL KNOWLEDGE ABOUT QUANTITIES

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Abstract
The concept of didactic transposition expands the analytical scope of educational research by including the transformations of knowledge orchestrated by diverse actors, from academic researchers to curriculum developers and teachers. Researchers adopt distinct positions regarding the institutions involved in this transposition process, fundamentally shaping which aspects of knowledge become questionable and which remain naturalised. Through examining three theoretical approaches—Stoffdidaktik, pedagogical content knowledge, and the anthropological theory of the didactic—this study illuminates how researcher positioning influences investigations of mathematical knowledge transformation. We use the mathematisation of quantities as an analytical case to demonstrate these distinctions, revealing how each position enables particular forms of epistemological inquiry whilst constraining others. This exploration establishes didactic transposition as a powerful meta-theoretical lens for delimiting analytical units across different perspectives, stimulating further research into their theoretical foundations. The analysis contributes to mathematics education research by providing a refined framework for understanding how epistemological assumptions embedded in researcher positioning influence both problem formulation and solution development in educational contexts.

Keywords: didactic transposition; theoretical perspectives; Stoffdidaktik; pedagogical content knowledge; anthropological theory of the didactic; numbers, quantities and measurement

1. INTRODUCTION

The transformation of mathematical knowledge from its scholarly origins to classroom manifestation represents a fundamental yet under-examined dimension of mathematics education research. Whilst considerable attention has focused on understanding student learning and teaching facilitation, the anterior processes through which mathematical knowledge undergoes selection, reconstruction, and legitimisation for educational purposes remain comparatively neglected. This epistemological gap carries profound implications for both research and practice, as our assumptions about the nature and origins of school

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mathematics fundamentally shape our approaches to investigating and addressing educational challenges.

This study addresses this critical lacuna by examining how researchers' positions within what Chevallard (1985) terms the 'didactic transposition process' influence their investigation of mathematical knowledge and its transformations. We argue that the institutional vantage point from which researchers observe and analyse these transformations—whether situated within scholarly mathematics, embedded in educational systems, or positioned externally to both—fundamentally determines which aspects of mathematical knowledge become available for questioning and which remain naturalised or invisible. This positioning constitutes a largely implicit yet powerfully determinative factor in shaping the scope, methods, and outcomes of mathematics education research.

To illuminate these theoretical distinctions, we examine the mathematical treatment of quantities and measurement as a particularly revealing case. The historical trajectory of this domain—including its near-elimination during the New Math reforms and subsequent problematic reintroduction—exemplifies how different epistemological positions lead researchers to formulate divergent problems and propose distinct solutions. Through this exploration, we demonstrate the utility of didactic transposition as a meta-theoretical lens for comparative analysis whilst contributing to mathematics education's development as a scientific discipline with increasingly sophisticated tools for reflexive examination of its theoretical foundations.

2. THE NOTION OF DIDACTIC TRANSPOSITION

2.1. Questioning the mathematical knowledge to be taught

The notion of didactic transposition, introduced by Yves Chevallard in the 1980s, fundamentally reconceptualises mathematics education research by insisting that the mathematical knowledge taught and learnt in schools cannot be taken as given but must itself become an object of critical inquiry (Chevallard, 1985; Chevallard & Bosch, 2020b). This epistemological shift redirects attention beyond immediate teaching and learning contexts to encompass the complex institutional processes through which mathematical knowledge undergoes deliberate transformation to ensure it becomes both 'teachable' and 'learnable'. The transposition process thus involves not only selecting mathematical concepts, problems, and strategies deemed necessary for education but also elaborating entire mathematical organisations that embed this knowledge in concrete practices suitable for school contexts.

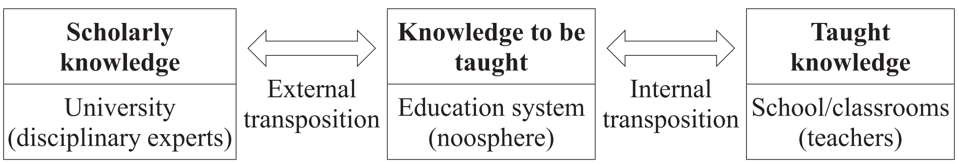

Figure 1. The process of didactic transposition and its elements (see Scheiner & Bosch, 2023)

To describe and analyse the didactic transposition process, several entities require careful delineation (see Figure 1). 'Scholarly knowledge' pertains to what educational systems identify as legitimate disciplinary knowledge for school study—knowledge typically developed by scholars in universities and research centres, among other professional institutions. Through processes of selection and transformation, certain elements of this scholarly knowledge become reconstituted as ‘knowledge to be taught’, which finds expression in curriculum guidelines and further elaboration in instructional proposals, textbooks, and teaching materials.

This selection and elaboration represent prolonged collective efforts undertaken by various actors constituting what the theory terms the *noosphere*—a sphere of influence encompassing curriculum developers, textbook authors, teacher associations, and scholars who mediate between research and educational practice. Within school systems, the ‘knowledge to be taught’ undergoes further transformation through teachers’ and students’ work, both outside and inside classrooms, to become ‘knowledge actually taught’. The ‘learnt knowledge’, whilst initially excluded from the theory’s original formulation, can also be explored as a crucial transposition component, recognising that this knowledge depends not solely on students but also on the specific circumstances and institutional constraints within which learning is actualised.

Several mathematics education studies, particularly those drawing upon the anthropological theory of the didactic (ATD, Chevallard & Bosch, 2020a), emphasise analysing the didactic transposition process as a crucial methodological element. This analytical approach proves instrumental in identifying phenomena affecting the teaching and learning of specific mathematical concepts, as well as the overall practice of mathematical activity within educational institutions. Examining didactic transposition begins with critical inquiries into the origin, nature, and purpose of imparted knowledge: What are ‘algebra’, ‘geometry’ or ‘proportionality’ as school subject areas made of, and why are these particular mathematical knowledge organisations included in curricula? How did these configurations emerge historically? In what various forms have they been represented in educational programmes over time, and what specific activities bring them to life in contemporary classroom settings?

By systematically questioning knowledge’s didactic transposition, educators and researchers are encouraged to denaturalise what might otherwise appear as inevitable features of school mathematics. This examination raises fundamental questions about why certain knowledge areas disappear from curricula, undergo transformation, or reappear in modified forms, and how these changes relate to persistent challenges in teaching and learning mathematics. Such an approach places epistemological inquiry—questions about the nature, origins, and purposes of taught and learnt knowledge—at the forefront of mathematics education research, thereby expanding the field’s analytical scope beyond immediate pedagogical concerns.

2.2. An extended unit of analysis

Analysing the didactic transposition process demands significant expansion of the unit of analysis traditionally considered in mathematics education research (Bosch & Gascón, 2006). Moving beyond immediate teacher-student interactions in classroom settings, researchers must trace mathematical knowledge’s genealogy and transformation across multiple institutions and extended timescales. This expanded perspective reveals how phenomena typically attributed to teaching difficulties or student learning obstacles may actually originate in much earlier transposition phases—in decisions about curriculum structure, in mathematical organisations presented in textbooks, or even in lacunae within scholarly mathematics itself when providing adequate foundations for educational purposes.

This methodological expansion enables crucial analytical distinctions between different types of problems emerging within mathematics education (Barquero et al., 2019). The ‘teaching problem’ concerns implementing given curricular content under specific classroom conditions: How to teach certain knowledge given particular institutional constraints? What difficulties arise in this implementation and how might they be addressed? The ‘curriculum problem’ operates differently, questioning the very selection and organisation of knowledge: What mathematical knowledge must be taught, to whom, when, and under what conditions? What challenges emerge from these selections and how might we respond to them? Beyond these lies the formulation of research problems maintaining critical distance from educational systems’ immediate concerns: Why is certain knowledge configured for teaching in particular curricular organisations and embedded in specific learning activities? What functions does this knowledge serve in current and historical scholarly mathematics? What alternative organisations might be possible?

Taking the didactic transposition process as the unit of analysis also facilitates systematic comparison of different theoretical approaches in mathematics education. By examining how various research frameworks integrate or position themselves relative to different agents, institutions, types of knowledge, and productions intervening in didactic transposition, we can better understand their complementary contributions and inherent limitations. In this paper, we raise the specific question of how different approaches delimit and position the relationship between school mathematics and university mathematics, aiming to illustrate such comparison whilst provoking new queries about the assumptions underlying these and other theoretical perspectives in the field.

3. SCHOOL AND SCHOLARLY MATHEMATICS: THREE THEORETICAL PERSPECTIVES

An example of such comparative analysis was presented in Scheiner and Bosch (2023), where we examined how three influential perspectives approach the relationship between school mathematics and scholarly mathematics: Klein’s elementary mathematics from a higher standpoint, Shulman’s transformation of disciplinary subject matter into subject matter for teaching, and Chevallard’s theory of didactic transposition. Whilst that analysis employed didactic transposition as a meta-theoretical perspective to map where other approaches could be located, the present study extends this work by using the theory to characterise the different conceptions of ‘knowledge’ assumed in each approach and the implications of these epistemological commitments.

3.1. Klein’s elementary mathematics from a higher standpoint

Felix Klein, whose influence on mathematics teacher education extended well beyond his native Germany, profoundly shaped how we conceive the relationship between advanced mathematics and school mathematics (Weigand et al., 2019). Through his seminal series ‘Elementary Mathematics from a Higher Standpoint’, Klein sought to address what he perceived as a problematic discontinuity between university-studied mathematics and school-taught mathematics. His objective was ensuring that prospective teachers could perceive deep continuities between these apparently disparate forms of mathematical knowledge, enabling them to view mathematics as a coherent, unified discipline rather than as disconnected educational levels.

Central to Klein's (1908/2016) approach was the notion of 'elementarisation' (Elementarisierung), which involves uncovering and articulating a mathematical domain's core essence rather than merely simplifying its content for pedagogical purposes (see Schubring, 2016). This method involves restructuring and clarifying foundational mathematical concepts to reveal what Klein termed the 'elementary'—not denoting simplicity or basic level but rather reflecting mathematics' fundamental structural aspects that should form the conceptual foundation of secondary mathematics education. This approach to understanding mathematics, which has become deeply rooted in the German tradition of subject-matter didactics (Stoffdidaktik), led to theoretical constructs such as 'fundamental ideas' (Grundideen) and 'basic mental models' (Grundvorstellungen). These concepts aim to capture mathematical domains' primary organising principles whilst offering meaningful interpretations that ensure deeper, more connected understanding of the subject.

3.2. Shulman's transformation of disciplinary subject matter into subject matter for teaching

Lee S. Shulman's contribution to educational thought introduced a conception of teacher knowledge that fundamentally challenged the traditional separation between content expertise and pedagogical skill. His notion of 'pedagogical content knowledge' (PCK) represents professional knowledge that goes beyond subject matter knowledge per se to encompass the specific understanding required for effective teaching (Shulman, 1986). Shulman defined pedagogical content knowledge as a distinctive amalgam of content and pedagogy, uniquely tailored to address diverse learners' needs, prior knowledge, and cognitive capabilities. In introducing this construct, Shulman bridged largely separate inquiry domains—subject matter scholarship and pedagogical research—thereby reshaping how educators and researchers approach both teacher education and the study of teaching as professional practice.

What emerged from Shulman's research program was the conviction that effective teaching requires teachers to 'transform' academic subject matter into forms that are pedagogically powerful yet maintain disciplinary integrity. The essence of this professional expertise lies in teachers' ability to reconstruct and represent subject matter in ways that resonate with students' existing knowledge and cognitive inclinations, ensuring that content remains both accessible to learners and true to its disciplinary foundations. This transformation involves selecting appropriate representations, identifying productive examples and analogies, recognising common misconceptions, and understanding the conceptual obstacles that students typically encounter within specific content domains.

3.3. Locating Klein and Shulman's position in the didactic transposition process

Using Chevallard's theory of didactic transposition as an analytical lens reveals distinct positioning of these approaches within the overall transformation process (see Scheiner & Bosch, 2023). Klein's approach emphasises the relationship between scholarly knowledge and knowledge to be taught, operating either within the scholarly institution itself or at the critical interface between the scholarly institution and the noosphere. The process of 'elementarisation' serves to connect 'scholarly knowledge' with the 'knowledge to be taught' through mathematical reconstruction rather than mere simplification. The distinctive aspect of Klein's approach lies in its treatment of both forms of knowledge as existing in reciprocal relationship: both are open to challenge, questioning, and reconstruction in light of educational needs and mathematical coherence. However, in Klein's vision, this transposition work is primarily undertaken by disciplinary experts—mathematicians with educational sensibilities—rather than by classroom teachers, who are expected to develop understanding of 'elementary mathematics from a higher standpoint' through their professional preparation.

In marked contrast, Shulman's approach focuses on individual teachers' crucial role in transforming scholarly knowledge (which he terms 'disciplinary knowledge' or 'subject matter content') into forms suitable for classroom teaching. Notably, Shulman's approach does not explicitly recognise an intermediate entity corresponding to the 'knowledge to be taught' in the transposition process. Instead, Shulman introduced pedagogical content knowledge as distinct professional knowledge representing teachers' understanding of how to make subject matter comprehensible to students. This knowledge, held by individual teachers and applied directly in classroom settings, encompasses understanding of student conceptions, productive representations, and effective instructional strategies. Reference to something approximating the knowledge to be taught can be found in what Shulman terms 'curricular knowledge'—understanding of the curricular programs and materials available for instruction—though this remains conceptually distinct from PCK itself (Shulman, 1986, p. 10).

4. ILLUSTRATING DIFFERENCES: THE CASE OF QUANTITIES AND MEASUREMENT

To further illuminate the theoretical and practical implications of these different positions within the didactic transposition process, we consider three research studies addressing the teaching and learning of quantities and measurement. This mathematical domain proves particularly revealing as it has undergone significant transformations in school curricula, including near-elimination during the New Math period and subsequent attempts at reintegration that continue to generate educational challenges.

4.1. Klein's approach and the 'comparational measurement theory'

Griesel's (2007) proposal for what he calls a 'comparational measurement theory' exemplifies how Klein's approach manifests in contemporary didactics of mathematics. Griesel initiated his work by critiquing the prevalent curricular treatment of number systems in German schools—the progression through natural numbers, integers, and rational numbers—particularly noting the problematic disconnection between numbers and quantities. He identified fundamental inadequacies in how curricula apply operations such as addition and multiplication across different number types, observing that whilst natural number operations are typically grounded in set-theoretic foundations, a unified operational framework linking numbers to quantities remains absent. This critique resonates deeply with Klein's emphasis on mathematical unity and coherence across educational levels.

Griesel extended his analysis to question even Frege's foundational work, arguing for the essential inclusion of measurement in any adequate conceptualisation of number. However, he identified a significant gap in the fundamental mathematical analysis of measurement itself, leaving the precise relationships between quantities, measurement procedures, and numerical representations inadequately theorised. In response, Griesel (2007) developed a theory that aims to provide unified mathematical foundations for introducing numbers and their operations through systematic emphasis on comprehensive quantities and quantity domains. His theory positions measuring as fundamentally involving multiplicative comparison of objects and quantity values (ratio-scales), with numbers emerging as the results of such comparisons. In this framework, additive comparisons and the composition of quantities emerge as secondary constructions, reversing the typical curricular sequence. Griesel explicitly acknowledged that his theoretical contribution requires further elaboration before effective school implementation: “This reform program may be accomplished only after developing a corresponding practical curriculum and its implementation in school practice” (Griesel, 2007, p. 37).

This example illustrates how the identified teaching problem is implicitly located in the realm of ‘knowledge to be taught’ yet is approached through developing new mathematical theory within scholarly mathematics. As Griesel (2007) states: “it is shown how Frege’s viewpoint of mathematics fundamentals [...] can be better implemented in the current construction of the system of real numbers in school” (p. 31). The researcher questions existing curricular mathematics, identifies its deficiencies regarding number construction and the marginalisation of quantity, then responds by elaborating new mathematical organisations that extend beyond existing scholarly productions. The distinction between ‘theoretical’ and ‘practical’ curriculum in Griesel’s formulation reveals his position at the interface between scholarly knowledge and knowledge to be taught, contributing to what we might term the external didactic transposition process.

4.2. Shulman’s approach and teachers’ pedagogical content knowledge about quantities

Research conducted within the pedagogical content knowledge (PCK) framework takes a markedly different approach to similar educational challenges. Passelaigue and Munier (2015) analysed French elementary school teachers’ knowledge about attributes (quantities) and measurement, focusing on both subject matter knowledge and pedagogical content knowledge. Their study addresses curricular recommendations that teachers engage students in direct comparison activities before introducing measurement procedures—a sequence intended to counter the common tendency to prioritise measurement techniques over conceptual understanding of physical attributes. The curriculum guidelines assume that teachers possess clear understanding of the conceptual distinctions between attributes and their measurement—an assumption the researchers set out to examine empirically.

The authors hypothesised that “when teaching attributes and measurement, teachers are not convinced of the utility of working specifically on attributes before (or independently of) measurement” (Passelaigue and Munier, 2015, p. 317). To investigate this hypothesis, they designed a word classification task requiring pre-service teachers to categorise terms such as ‘length’, ‘volume’, ‘comparison’, ‘evidence’, ‘instrument’, ‘gram’, ‘unit’, ‘precision’, and ‘number’. The teachers’ classifications were then compared to what the researchers termed ‘expert answers’, revealing significant discrepancies in conceptual understanding.

The study confirmed that concepts of attributes and measurement are indeed poorly understood by pre-service elementary school teachers. Whilst participants might grasp specific instances such as length or mass, comprehensive understanding of attributes and measurement as general concepts—deemed crucial for effective teaching—often remained elusive. The authors report “an erroneous understanding of the concept of attribute, one that is far removed from the scientific definition” and observe that “for a large percentage of our teacher trainees, an attribute is something vague, ill-defined, not very precise” (p. 332). Their conclusions advocate for enhanced development of elementary teachers' specialised content knowledge about attributes and measurement, alongside dissemination of teaching approaches that emphasise attribute comparison without measurement, thereby contributing to pedagogical content knowledge development.

This research exemplifies how studies positioned within the PCK framework tend to treat the knowledge to be taught as given, focusing instead on perceived deficiencies in teachers' understanding of this predetermined content. The external didactic transposition process—including the historical absence of adequate theorisation of quantities in school mathematics—remains unexamined. The solution is formulated in terms of enhanced teacher education rather than questioning the epistemological infrastructure available for such education. The positioning near the noosphere accepts existing categorisations whilst attributing difficulties to individual knowledge deficits rather than systemic epistemological gaps.

4.3. The didactic transposition of quantities and measurement

Research approaching these issues through the lens of didactic transposition adopts a fundamentally different stance, as illustrated by Chambris (2017). Her research question is formulated explicitly in terms of the didactic transposition process: What place and role have quantities played in French mathematics education before, during, and after the New Math reform? Through historical analyses, Chambris traces the evolution of teaching about numbers and ratios since the early twentieth century, documenting the virtual disappearance of quantities from curricula during the New Math period and their inadequate reintroduction in subsequent reforms.

Chambris' (2017) analysis reveals how the absence of theoretical foundations for quantities in school mathematics created lasting impediments to teaching numbers and ratios effectively. She observes:

> Before the New Math, whole numbers, place value, and operations were taught with both discrete and continuous quantities. The creation of the 'measurement' domain is the visible side of the transposition of the set theory and the elimination of continuous quantities in the reference [scholarly] knowledge for numbers, operations and ratios. Yet, continuous quantities seem to be a key input for conceptualization of numbers and ratios […] and the epistemology […] of numbers also fosters the desirable approaches on continuous quantities for numbers and ratios. (p. 137)

The author explicitly references Griesel's (2007) work as exemplifying the kind of scholarly mathematical development about quantities that remains absent from the epistemological infrastructure supporting school mathematics. She analyses the teaching and learning difficulties created by this absence, particularly in the construction, interpretation, and use of whole number operations when reference to concrete quantities is eliminated. Chambris concludes by emphasising the importance of understanding curriculum reforms' long-term effects and, in the French case, the persistent lack of adequate reference scholarly knowledge for numbers and ratios. Her analysis shows how the reintroduction of quantities 'for didactical needs' through creating a 'measurement' domain failed to provide sufficiently robust mathematical foundations for relating quantities to numerical concepts.

This approach positions the researcher outside the institutions directly involved in the didactic transposition process, enabling critical examination of the entire transformation chain. Taking the knowledge to be taught as the primary object of study, Chambris analyses its historical evolution and variable relationships with scholarly knowledge, highlighting the absence of adequate quantity theory and tracing the cascading effects of this epistemological gap. Her analysis employs sophisticated concepts from the theory of didactic transposition, including ecological metaphors of knowledge ‘habitats’, ‘niches’, and ‘trophic chains’, to map the complex interdependencies within mathematical knowledge systems.

4.4. Comparative synthesis

These three studies, whilst addressing related difficulties in teaching and learning about quantities and measurement, reveal how researcher positioning within the didactic transposition process fundamentally shapes problem formulation and proposed solutions. Griesel (2007), working within Klein’s tradition, identifies curricular inadequacies but responds by developing new scholarly mathematics, explicitly acknowledging the need for subsequent transposition into ‘practical curriculum’. His position at the interface between scholarly and curricular knowledge enables him to treat both as variables requiring optimisation. Passelaigue and Munier (2015), operating within Shulman’s framework, locate the problem in teachers’ insufficient knowledge of predetermined content, proposing enhanced teacher education as the solution whilst leaving unexamined the epistemological infrastructure that should support such education. Chambris (2017), adopting an external analytical position, reveals how the absence of adequate scholarly foundations for quantities created systemic problems that persist despite attempts at curricular repair, demonstrating how epistemological gaps propagate through the entire transposition chain.

5. DIDACTIC TRANSPOSITION AS A QUESTIONING TOOL

5.1. The position of the researcher in the didactic transposition process

Our comparative analysis reveals that rather than simply identifying analogous elements within the didactic transposition process across different approaches, greater insight emerges from discerning which entities each approach treats as open to questioning versus which are accepted as given. Using a mathematical analogy, we can identify which aspects function as ‘variables’ subject to modification and which operate as ‘constants’ with fixed values. This analytical strategy illuminates both the affordances and constraints inherent in different researcher positions.

Klein’s approach implies comprehensive transformation of both ‘scholarly knowledge’ and its relationship to ‘knowledge to be taught’ through developing new mathematical theorisations that can ground curriculum proposals. Research in this tradition begins with fundamental questions about what mathematics is possible and appropriate for teaching at various educational levels, then develops specific bodies of mathematical knowledge to address identified problems and provide needed infrastructure. The resulting productions may be located within scholarly institutions or the noosphere, but crucially, neither existing scholarly mathematics nor current curricular mathematics is taken as given—both function as variables in our analogy. Griesel’s (2007) ‘comparational measurement theory’ exemplifies this position through its willingness to extend beyond Frege’s foundations to create new mathematical frameworks specifically oriented toward educational needs.

Conversely, Shulman's approach emphasises the emergence of pedagogical content knowledge as distinctive professional understanding that enables teachers to transform subject matter for instruction. Yet this framework does not extend to questioning the foundations of subject matter content itself, nor does it examine how the knowledge to be taught comes to be delimited and structured. Within this approach, the epistemological infrastructure supporting both content knowledge and pedagogical content knowledge remains unexamined. Passelaigue and Munier's (2015) study exemplifies this constraint: teachers' knowledge about quantities and measurement is analysed without reference to the absence of adequate quantity theory in either scholarly or school mathematics. Their recommendation for expanded teacher education in 'concepts of attribute and measurement' presumes these concepts exist in adequate form, requiring only better transmission rather than fundamental reconstruction.

The pivotal distinction concerns the institutional position adopted by researchers relative to the transposition process. Klein's approach, located within the external transposition, develops new forms and foundations for the knowledge to be taught and critically examines how it relates to scholarly knowledge, even proposing enhancements to the latter to better support the former. Shulman's approach, positioned within the school system or its noosphere, treats both scholarly knowledge and curricular knowledge as 'constants' (or 'givens') whilst focusing on the personal knowledge teachers require for effective internal transposition. In this framework, scholarly knowledge and knowledge to be taught often merge into a single entity called 'pedagogical content knowledge', with research problematising its transformation rather than its constitution. Developments of Shulman's approach such as the one proposed by Ball, Thames and Phelps (2008), identify different forms of mathematical knowledge for teaching, including three elements in the 'subject matter knowledge' (common content, mathematical horizon and specialised or advanced content) and in the 'pedagogical content knowledge' (content and students, content and teaching, curriculum). However, as pointed out by Winsløw (2017), who also compares the three approaches considered here, this development does not expand the unit of analysis in terms of the didactic transposition process. In particular, it does not introduce "any institutional perspective and, consequently, of any explicit distinction between university and school mathematics" (p. 79). The methodological consequences are significant: no tools are provided for deconstructing and reconstructing content knowledge itself, and research tends to individualise what may be systemic epistemological problems.

Neither approach questions the entire transposition process, a perspective that requires researchers to adopt a more external position regarding both school and scholarly institutions. This external positioning, exemplified in Chevallard's approach and Chambris' (2017) research, enables comprehensive analysis of how knowledge transforms across institutional boundaries whilst revealing the often-invisible work of the noosphere in shaping what counts as legitimate school mathematics.

5.2. The need for epistemological models of reference

The exploration of didactic transposition processes requires researchers to develop sophisticated understanding of the knowledge organisations constituting the knowledge to be taught, the taught knowledge derived from it, and the various scholarly knowledge organisations that legitimate the teaching process. Within this complex institutional landscape, it becomes crucial for researchers to explicate their assumptions regarding the components and functions of these different bodies of knowledge. Consequently, the study of didactic transposition processes is intimately linked with the development of what are termed epistemological models of reference—explicit theoretical constructions that clarify researchers' positions regarding various forms of mathematical knowledge whilst preventing unconscious adoption of perspectives inherent to the institutions under study (see Figure 2).

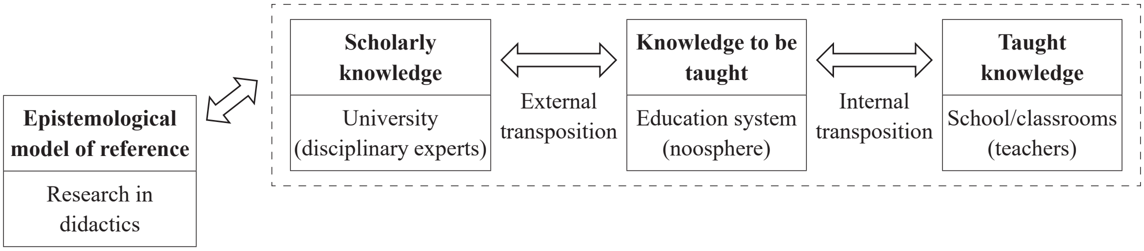

Figure 2. The external position of researchers (based on Chevallard & Bosch, 2020b)

These epistemological models serve multiple essential functions in didactic research. As analytical tools, they provide explicit frameworks for comparing knowledge organisations across different institutions and historical periods. As reflexive instruments, they help researchers maintain awareness of their own epistemological assumptions and positioning: how they conceive the knowledge at play, what elements they take for granted, which ones are questioned, etc. As generative devices, they suggest alternative knowledge organisations beyond those currently institutionalised, thereby opening new possibilities for curriculum design and pedagogical innovation. The elaboration of such models, always provisional and subject to refinement based on empirical findings and theoretical developments, constitutes both a methodological necessity and a theoretical contribution to the field (Bosch, 2015; Florensa et al., 2020; Gascón, 2024; Ruiz-Munzón et al., 2013).

Without explicit formulation of epistemological models of reference, researchers risk unconsciously adopting the prevailing conceptions and implicit assumptions of the institutions they study, whether these derive from scholarly mathematics, the noosphere, or school mathematics itself. This unconscious adoption hinders critical examination of how mathematical knowledge is constituted, selected, and transformed for educational purposes.

Regarding the specific challenges of teaching quantities and measurement, the elaboration of reference models may take various forms depending on the research questions addressed. If we consider the three previous works of Griesel (2007), Passelaigue and Munier (2015), and Chambris and Visnovska (2022), we can observe how all authors develop, more or less explicitly, their own definition, conception or epistemological model about quantities, assigning different roles to these reference models in their research.

The most explicit construction is the one of Griesel (2007), whose work can be directly interpreted as the development of a reference epistemological model within (or close to) the scholarly institution for educational purposes, that is, to serve as a guide for further elaborations of the knowledge to be taught organisation. Despite the formal proximity of Griesel's model to scholarly mathematics, it is important to notice its novelty and detachment from the scholarly institution, as it provides a totally new or alternative construction compared to what already exists.

Even if Passelaigue and Munier (2015) remain close to the French noosphere, they rely on a sophisticated reference model developed in Munier and Passelaigue (2012) for the role of quantities and measurement in science education. It is very interesting to see how the change of institutional perspective—from mathematics to sciences—provides the reader with a particular form of detachment and points at a shortfall of the school knowledge organisations that exceeds the case of mathematics. In their approach, the authors conclude about the vagueness in which teachers are left to address this issue with pupils. As a consequence, the proposed reference model is aimed at developing teacher education proposals, answering what the authors see as "an epistemological training about these concepts and a didactic training to reflect on the way they can be worked with the students" (Munier & Passelaigue, 2012, p. 29, our translation).

Finally, Chambris (2017) and Chambris and Visnovska (2022) developed a local reference model based on classic mathematical treatises in arithmetic from the 18th and 19th centuries about concrete and abstract numbers, quantities, numeration, numbers and operations. They use this model to analyse the teaching of numbers, ratios and proportionality in the evolution of school mathematics, pointing at the effects of the historical marginalisation of quantities, ratios and units. It is clear, in this case, that the reference model appears as an analytical tool to identify didactic phenomena and point at the broader question of what mathematical resources might adequately support the mathematisation of quantities at school.

The construction of epistemological models of reference can thus be used more normatively to propose direct interventions to the didactic transposition process, whether in the external transposition (Griesel) or in the teacher education proposals organised within the noosphere (Passelaigue & Munier). They can also serve as a research tool for more descriptive and analytical purposes (Chambris & Visnovska). In all cases, they contribute to the collective development of mathematical and didactic resources needed for educational improvement. By questioning the nature of the knowledge proposed for school teaching and analysing how this knowledge transforms into classroom activities, didactic research provides educators with enhanced frameworks for analysing, designing, implementing, and assessing teaching and learning processes. Ultimately, the systematic analysis of didactic transposition leads to the creation of conceptual and operational tools that can support its ongoing evolution and refinement.

6. CONCLUSION

This paper has demonstrated that the position researchers adopt within the didactic transposition process profoundly shapes their approach to investigating mathematical knowledge and its educational transformations. Through examination of Klein's, Shulman's, and Chevallard's frameworks, we have illuminated how different institutional positions enable distinct forms of epistemological inquiry whilst simultaneously constraining others. The case of quantities and measurement has served to ground these theoretical distinctions empirically, revealing how researcher positioning influences not only problem formulation and methodological choices but also the very conception of what constitutes a solution to educational challenges.

Our analysis contributes to mathematics education research by establishing didactic transposition as a meta-theoretical lens for comparative analysis. By making explicit the usually implicit positioning of researchers relative to the institutions involved in knowledge transformation, this framework enables more precise theoretical comparisons and opens possibilities for strategic choices about where and how to intervene in educational systems. The development and use of epistemological models of reference emerges as crucial for maintaining the critical distance necessary for comprehensive analysis, whilst also contributing to the mathematical and didactic infrastructure needed for educational progress.

The implications extend beyond theoretical clarification to encompass practical consequences for curriculum development, teacher education, and educational policy. Recognition that persistent challenges in mathematics education may stem from systemic epistemological gaps rather than local implementation problems suggests the need for coordinated efforts across the entire transposition process. As mathematics education continues to mature as a scientific discipline, explicit consideration of researcher positioning and its epistemological implications becomes increasingly vital for both theoretical advancement and practical innovation. The framework developed here provides tools for such reflexive analysis, facilitating dialogue across different research traditions whilst maintaining awareness of their complementary contributions and inherent limitations.

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